Longest Common Subsequence

E-OLYMP **1618. The longest Common Subsequence** Two sequences of integers are given. Find the length of their longest common subsequence (the subsequence is a sequence that can be derived from another sequence by deleting some elements without changing the order of the remaining elements).

► A ***subsequence*** of a sequence is a set of elements that appear in left-to-right order, but not necessarily consecutively. A subsequence can be derived from a sequence only by deletion of some elements.

For example, consider the sequence {2, 1, 3, 5}. Then

- {1, 5}, {2}, {2, 3, 5} are subsequences;
- {5, 1}, {2, 3, 1} are not subsequences;

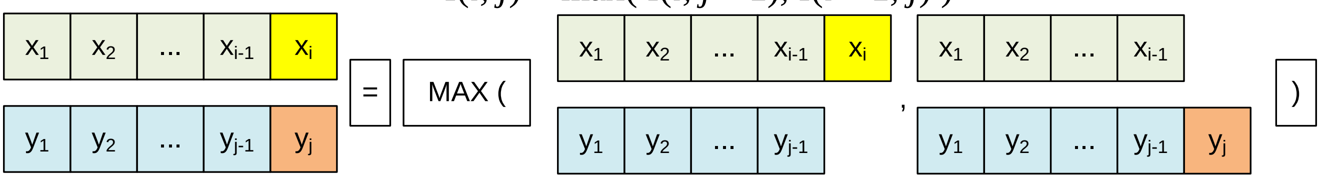
A ***common subequence*** of two sequences is a subsequence that appears in both sequences. A ***longest common subequence*** (***lcs***) is a common subsequence of maximal length.

For example, the longest common subequence of {1, 2, 3} and {2, 1, 3, 5} can be {1, 3} or {2, 3}. The length of lcs is 2.

Let f(i, j) be the length of the longest common subsequence of sequences $x_1x_2\ldots x_i$ and $y_1y_2\ldots y_j$.

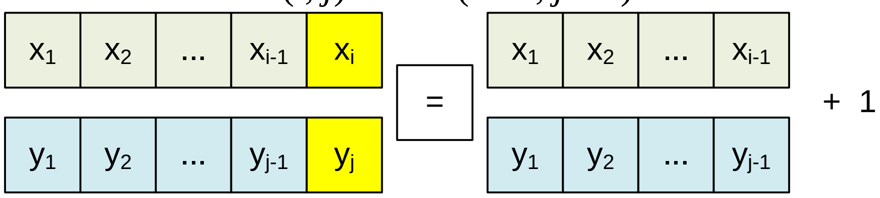
If $x_i \neq y_j$, then we find ***lcs*** among $x_1x_2\ldots x_i$ and $y_1y_2\ldots y_{j-1}$, and also among $x_1x_2\ldots x_{i-1}$ and $y_1y_2\ldots y_j$. Return the biggest value:

$$f(i, j) = \max(\ f(i, j-1), f(i-1, j)\ )$$



If $x_i = y_j$, then we find ***lcs*** among $x_1x_2\ldots x_{i-1}$ and $y_1y_2\ldots y_{j-1}$:

$$f(i, j) = 1 + f(i-1, j-1)$$



If one of the sequences is empty, then their lcs is empty:

$$f(0, j) = f(i, 0) = 0$$

Let's summarize the recurrence relation:

$$f(i,j) = \begin{cases} \max(f(i, j-1), f(i-1, j)), x_i \neq y_j \\ f(i-1, j-1)+1, x_i = y_j \\ 0, i = 0 \; or \; j = 0 \end{cases}$$

The values f(i, j) will be stored in array m[0..1000, 0..1000], where m[0][i] = m[i][0] = 0. Each next line of array m[i][j] is calculated through the previous one. Therefore, to find the answer, it is enough to keep in memory only two lines of length 1000.

Let X = *abcdgh*, Y = *aedfhr*. The the longest common subsequence is *adh*, its length equals to f(6, 6) = 3.

f(i, j)		X	a	b	c	d	g	h
		0	1	2	3	4	5	6
Y	0	0	0	0	0	0	0	0
a	1	0	1(a)	1	1	1	1	1
e	2	0	1	1	1	1	1	1
d	3	0	1	1	1	2(d)	2	2
f	4	0	1	1	1	2	2	2
h	5	0	1	1	1	2	2	3(h)
r	6	0	1	1	1	2	2	3

f(6, 6) = max(f(6, 5), f(5, 6)) = max(2, 3) = 3, because Y[6] = $r \neq h$ = X[6].
f(5, 6) = 1 + f(4, 5) = 1 + 2 = 3, because Y[5] = h = X[6].

Arrays x and y contain input sequences, n and m are their lengths. Array mas contains two last lines of dynamic calculations.

```
#define SIZE 1010
int x[SIZE], y[SIZE], mas[2][SIZE];
```

Main part of the program. Read input sequences to arrays, starting from the first index. Then read the data into x[1..n] and y[1..m].

```
scanf("%d",&n);
for(i = 1; i <= n; i++) scanf("%d",&x[i]);
scanf("%d",&m);
for(i = 1; i <= m; i++) scanf("%d",&y[i]);
```

Fill array mas with zeroes. Dynamically calculate the values f(i, j). Initially mas[0][j] contains the values f(0, j). Then put into mas[1][j] the values f(1, j). Since to calculate f(2, j) it is enough to have the values of the previous row of mas array, the values of f(2, j) can be stored in mas [0][j], the values of f(3, j) in mas [1][j] and so on.

```
memset(mas,0,sizeof(mas));
```

```
for(i = 1; i <= n; i++)
for(j = 1; j <= m; j++)
  if (x[i] == y[j])
    mas[i%2][j] = 1 + mas[(i+1)%2][j-1];
  else
    mas[i%2][j] = max(mas[(i+1)%2][j],mas[i%2][j-1]);
```

Print the answer, that is located in the cell mas[n][m]. Take the first argument modulo 2.

```
printf("%d\n",mas[n%2][m]);
```

E-OLYMP **1079. Removing the letters** You are given two words (each word consists of upper-case English letters). Delete some letters from each word so that the resulting words become equal.

Find the maximum possible length of the resulting word.

► The answer to the problem is the length of the ***longest common subsequence*** (**LCS**) of input sequences of uppercase Latin letters.

Let f(i, j) be the longest common subsequence of sequences $x_1x_2\ldots x_i$ and $y_1y_2\ldots y_j$.

If $x_i \neq y_j$, then find LCS between $x_1x_2\ldots x_{i-1}$ and $y_1y_2\ldots y_j$, and also between $x_1x_2\ldots x_i$ and $y_1y_2\ldots y_{j-1}$. Return the largest of them:

$$f(i, j) = \max(\, f(i, j-1), f(i-1, j)\,)$$

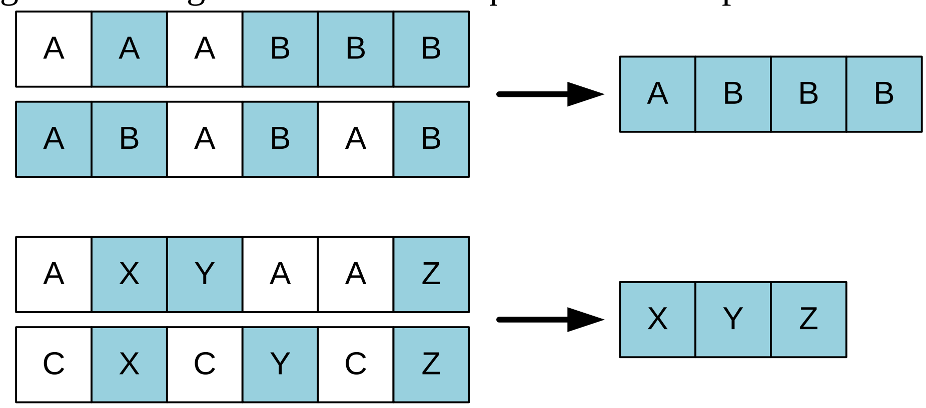
If $x_i = y_j$, then find LCS between $x_1x_2\ldots x_{i-1}$ and $y_1y_2\ldots y_{j-1}$:

$$f(i, j) = 1 + f(i-1, j-1)$$

The values f(i, j) will be stored in array m[0.. SIZE, 0.. SIZE], where m[0][i] = m[i][0] = 0. Since the length of words is no more than 200 characters then assign SIZE = 201.

Each next line of array m[i][j] is calculated through the previous one. Therefore, to find the answer, it is enough to keep only two lines in memory.

Here is given the largest common subsequences for samples.



E-OLYMP **4260. LCS - 2** Two strings are given. Find and print their longest common subsequence.

► In the problem you must find the largest common subsequence (LCS) of two strings and print it.

Construct an array dp, where dp[i][j] is the length of LCS of strings $x_{[1\ldots i]}$ and $y_{[1\ldots j]}$. The value of dp[n][m] equals to the length of LCS of input strings ($|x| = n$, $|y| = m$). Move through the matrix dp from the cell (n, m) to (0, 0). For the current position (i, j) we have:

- If symbols x_i and y_j are the same, then this character must be present in the LCS, store it into the resulting string *res*. Move in array dp from cell (i, j) to cell (i – 1, j – 1), that is, then construct LCS ($x_{[1\ldots i-1]}$, $y_{[1\ldots j-1]}$).
- If symbols x_i and y_j are different, then we can move in array dp from cell (i, j) either to cell (i, j – 1) or to cell (i – 1, j). Since the largest subsequence is being built, the transition should be made to the cell where the value is greater. If dp[i – 1][j] = dp[i][j – 1], we can go to any of the specified cells.

Find the LCS for two string given in a sample.

dp[i][j]		X	a	b	a	c	a	b	a
		0	1	2	3	4	5	6	7
Y	0	0	0	0	0	0	0	0	0
d	1	0	0	0	0	0	0	0	0
a	2	0	1	1	1	1	1	1	1
c	3	0	1	1	1	2	2	2	2
a	4	0	1	1	2	2	3	3	3
b	5	0	1	2	2	2	2	4	4
c	6	0	1	2	2	3	3	4	4

We start to search the LCS from position (i, j) = (6, 7).

y[6] ≠ x[7], move to any adjacent cell with the maximum value. For example to (5, 7).

y[5] ≠ x[7], move to (5, 6).

y[5] = x[6] = ‘b’, move diagonally, include letter ‘b’ to LCS.

Declare the inut strings x and y. To find their LCS declare dp array.

```
#define MAX 1001
int dp[MAX][MAX];
string x, y, res;
```

Read the input lines. Add a space to them so that the indexing will start from 1.

```
cin >> x; n = x.length(); x = " " + x;
cin >> y; m = y.length(); y = " " + y;
```

Finding the longest common subsequence.

```
for (i = 1; i <= n; i++)
for (j = 1; j <= m; j++)
  if (x[i] == y[j])
    dp[i][j] = dp[i - 1][j - 1] + 1;
  else
    dp[i][j] = max(dp[i - 1][j], dp[i][j - 1]);
```

Construct the LCS in the string *res*.

```
i = n; j = m;
while (i >= 1 && j >= 1)
  if (x[i] == y[j])
  {
    res = res + x[i];
    i--; j--;
  }
  else
  {
    if (dp[i - 1][j] > dp[i][j - 1])
      i--;
    else
      j--;
  }
```

Invert and print the resulting string.

```
reverse(res.begin(), res.end());
cout << res << endl;
```

E-OLYMP **1765. Three sequences** Three sequences of integers are given. Find the length of their longest common subsequence.

► Let a, b, c be three input sequences. Let f(i, j, k) be the length of LCS of sequences a[1..i], b[1..j] and c[1..k]. The value f(i, j, k) we shall keep in dp[i][j][k].

If a[i] = b[j] = c[k], then

$$f(i, j, k) = 1 + f(i - 1, j - 1, k - 1)$$

Otherwise

$$f(i, j, k) = \max(\ f(i - 1, j, k), f(i, j - 1, k), f(i, j, k - 1)\ )$$